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LETTER TO THE EDITOR

New derivation of a conserved quantity for Lagrangian systems

M Lutzky
10111 Quinby St, Silver Spring, MD 20901, USA

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Abstract. A new derivation of a recently discussed conserved quantity is given using a general procedure which determines N constants of the motion for any N-dimensional system possessing a non-Noether symmetry.

It has recently been established that any non-Noether symmetry of a dynamical system with Lagrangian $L(q, \dot{q}, t)$ determines a constant of the motion given by

$$\phi = \frac{E(D)}{D} + \frac{\partial \eta^l}{\partial q^l} + \frac{\partial \dot{\eta}^l}{\partial \dot{q}^l} \tag{1}$$

where $D = \left| \frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^l} \right|$ and $E = \eta^l(q, \dot{q}, t)\frac{\partial}{\partial q^l} + \dot{\eta}^l(q, \dot{q}, t)\frac{\partial}{\partial \dot{q}^l}$ generates the symmetry [1]. This was demonstrated by making use of a conservation law found by Hojman [2] and later generalized by Gonzalez-Gascon [3]. In this note we utilize methods of differential geometry to give a more direct proof, and in particular we show that the result follows from a general procedure which allows the derivation of N conserved quantities for any N-dimensional system possessing a non-Noether symmetry.

We define the Cartan one-form $\theta = \left(L - \frac{\partial L}{\partial \dot{q}^l}\right) \mathrm{d}t - \frac{\partial L}{\partial \dot{q}^l}\,\mathrm{d}q^l$ and its Lie derivative $\bar{\theta} = \mathcal{L}_E \theta$. Using these we construct N volume forms $\Omega_k = \mathrm{d}t\,\mathrm{d}\bar{\theta}\,\mathrm{d}\bar{\theta} \ldots \mathrm{d}\theta\,\mathrm{d}\theta$, $k = 1, 2, \ldots N$, where each Ω_k contains k factors of $\mathrm{d}\bar{\theta}$ and $N-k$ factors of $\mathrm{d}\theta$. Each Ω_k may be expressed in the form $\Omega_k = \rho_k\,\mathrm{d}t\,\mathrm{d}q^1 \ldots dq^N d\dot{q}^1 \ldots d\dot{q}^N$ and it has been shown in [4] that every $\frac{\rho_k}{D}$ is a constant of the motion. (Strictly speaking, the proof given in [4] is valid only if θ and $\bar{\theta}$ are both Cartan forms, so that there exist Lagrangians L and $\bar{L}$ which lead to the same equations of motion. Essentially the same proof, however, continues to hold as long as $\bar{\theta} = \mathcal{L}_E \theta$, where θ is a Cartan form, but $\bar{\theta}$ need not be). The ρ_k may be evaluated by expressing $\mathrm{d}\theta, \mathrm{d}\bar{\theta}$ in terms of the time, coordinate and velocity differentials in the original expression for Ω_k; in the particular case of ρ_1, however, a shorter method exists which leads to (1).

We define $v = \mathrm{d}t\,\mathrm{d}q^1 \ldots \mathrm{d}q^N\,\mathrm{d}\dot{q}^1 \ldots \mathrm{d}\dot{q}^N$, and consider the identity

$$\mathrm{d}t\,\mathrm{d}\theta \ldots \mathrm{d}\theta = (-1)^{N+1} N! D v \tag{2}$$

where the left-hand side contains N factors of $\mathrm{d}\theta$. Applying the Lie derivative operator $\mathcal{L}_E$ to the left-hand side of (2) yields $N\,\mathrm{d}t\,\mathrm{d}\bar{\theta}\,\mathrm{d}\theta \ldots \mathrm{d}\theta = N\rho_1 v$, where the first term contains $N-1$ factors of $\mathrm{d}\theta$.

Applying the same operator to the right-hand side of (2) gives

$$(-1)^{N+1}N!\left[E\{D\}+\left\{\frac{\partial\eta^l}{\partial q^l}+\frac{\partial\dot{\eta}^l}{\partial\dot{q}^l}\right\}D\right]v.$$

Equating these two results and referring to (1) then gives

$$\frac{\rho_1}{D}=(-1)^{N+1}(N-1)!\phi.$$

Since it is known that $\frac{\rho_1}{D}$ is conserved, we see that ϕ is also conserved, which establishes the desired result.

References

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